Unified Modeling of Corporate Debt, Credit Derivatives, and Equity Derivatives

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Overview and Main Themes

- Traditional models in asset pricing and dynamics governing asset prices do not incorporate default considerations. Could be considered a flaw.
- This is true in (i) term structure of defaultable debt, (ii) convertible bond pricing, and (iii) individual equity options.

Link Between Put Options and Credit Default Swaps

- There is clearly a link!.
- However, the argument must be tempered by the fact that most transaction volume for individual equity options is concentrated in short-maturity options and near money options. Is it necessary to incorporate default over short-term? Are default-free dynamics good enough approximation?
- Default may introduce additional risk-neutral skewness in individual equity return distributions. However, the risk-neutral skewness is much smaller in individual names compared to the S&P index [Bakshi, Kapadia, and Madan (RFS 2003)]. May be more relevant for names with high systematic risk.
- What is the precise contribution of default to risk-neutral volatility and risk-neutral skewness [see Bakshi and Madan (Management Science, January 2007)]. Some empirical and theoretical characterizations are necessary to tease out the impact of default.
- Summary: There is a far more **compelling** need to incorporate default in fixed income contingent claims than in equities, especially at short-horizons.

Basic Starting Point

- Model the *pre-default* stock dynamics under an EMM $\mathbb{Q}$ as a one-dimensional diffusion:

$$dS_t = [r(t) - q(t) + \lambda(S_t, t)]S_t\, dt + \sigma(S_t, t)S_t\, dB_t, \;\; S_0 = S > 0,$$

  r, q, σ and λ are the short rate, dividend yield, volatility, and default intensity.

- If the diffusion can hit zero, they kill it at the first hitting time of zero, T_0, and send it to a **cemetery (bankruptcy) state** Δ, where it remains forever.
- ***Jump-to-default*** arrives at the first jump time $\tilde{\zeta}$ of a doubly-stochastic Poisson process with intensity $\lambda(S_t, t)$. The **time of default** is $\zeta = \min\{T_0, \tilde{\zeta}\}$.
- Assume stock holders do not receive any recovery in the event of default.
- Addition of λ in the drift $r - q + \lambda$ compensates for default to insure that the discounted gain process to the stock holders is a martingale under the EMM.

Corporate Bonds

- The time-t price of a **defaultable zero-coupon bond** with face value of \$1 and no recovery in default:

$$B(S,t;T) = e^{-\int_t^T r(u)du} Q(S,t;T),$$

  where the **(risk-neutral) survival probability** is:

$$Q(S,t;T) = \mathbb{E}[e^{-\int_t^T \lambda(S_u,u)du} \mathbf{1}_{\{T_0 > T\}} | S_t = S].$$

- In the above framework, they assumes that interest rate is deterministic. Needs to be refined for fixed income claims (less of an issue for equity claims).
- The intensity function $\lambda(S_t,t)$ depends only on S_t.

A Tractable Class of Stock Price Processes

- **Suppose *pre-default* stock price dynamics:**

$$dS_t = [r - q + \lambda(S_t)]S_t\,dt + \sigma S_t\,dB_t,\ S_0 = S > 0,$$
$$\lambda(S) = \frac{\alpha}{S^p},\ \alpha > 0,\ p > 0.$$

- Constant σ.
- This process cannot diffuse to zero. **Time of default** ζ is the first jump time of a doubly stochastic Poisson process with intensity $\lambda(S)$.
- $\lambda(S) \to \infty$ as $S \to 0$, making default inevitable at low stock prices.
- **Obtain closed-form solutions in this model (V.L., "Pricing Equity Derivatives subject to Bankruptcy," *Mathematical Finance*, 2006, 16 (2), 255-282.**
- <u>Question:</u> Why are the desirable features of restricting intensities to the class: $\lambda(S) = \frac{\alpha}{S^p}$?

Innovations in Solution Techniques

- Computing expectations of the form:

$$V_\Psi(S,T) = e^{-rT}\mathbb{E}\left[e^{-\int_0^T \lambda(S_t)dt}\Psi(S_T)\right].$$

- $e^{-\int_0^T \lambda(S_t)dt}$ can be removed by **changing measure via Girsanov**:

$$V_\Psi(S,T) = e^{-qT}S\widehat{\mathbb{E}}\left[S_T^{-1}\Psi(S_T)\right],$$

$\widehat{\mathbb{E}}$ is w.r.t. $\widehat{\mathbb{Q}}$ under which $\widehat{B}_t := B_t - \sigma t$ is a standard BM and

$$dS_t = (r - q + \sigma^2 + \alpha S_t^{-p})S_t\, dt + \sigma S_t\, d\widehat{B}_t,\ S_0 = S > 0.$$

- The pre-default stock process under $\hat{Q}$ can be represented as:

$$S_t = (\beta^{-1}X^{(\nu)}_{\tau(t)})^{\frac{1}{p}},$$

where X is a diffusion process

$$dX_t = [2(\nu+1)X_t \mathbf{+1}]dt + 2X_t dW_t,\ X_0 = x = \beta S^p,$$

$$\beta := p\sigma^2/(4\alpha),\ \nu := 2(r - q + \sigma^2/2)/(p\sigma^2),\ \tau(t) := p^2\sigma^2 t/4.$$

Representation in Terms of Asian Options

- With the transformations, the problem reduces to computing

$$V_\Psi(S,T) = e^{-qT} S E_x^{(\nu)}[(X_\tau/\beta)^{-\frac{1}{p}} \Psi((X_\tau/\beta)^{\frac{1}{p}})],$$

  where $E_x^{(\nu)}$ is w.r.t. the probability law of X starting at $x = \beta S^p$.

- The process X is closely related to the problem of pricing **Asian options** (Geman and Yor (1993), Donati-Martin and Yor (2001), Linetsky (2004)).

- The spectral expansion of the transition density of X is available in closed form, yielding closed-form pricing formulas for corporate bonds and stock options in the form of spectral expansions.

Other Extensions for Intensity Processes

- Alternative intensity specification:

$$\lambda(S) = \frac{c}{\ln(S/\mathcal{B})}, \; c > 0, \; \mathcal{B} > 0, \; S > \mathcal{B}.$$

  This specification is similar to the one used in Madan and Unal (1998).

Model with CEV Variance

- Pre-default stock dynamics:

$$dS_t = [r(t) - q(t) + \lambda(S_t,t)]S_t\,dt + \sigma(S_t,t)S_t\,dB_t,\ S_0 = S > 0.$$

- To be consistent with the **leverage effect**, **constant elasticity of variance (CEV)** volatility specification is also adopted:

$$\sigma(S,t) = a(t)S^{\beta},$$

  $\beta < 0$ is the **volatility elasticity** and $a(t) > 0$ is the (time-dependent) **volatility scale parameter**.

- To be consistent with the evidence linking credit spreads to stock price volatility, **default intensity** — affine function of the instantaneous variance of the stock:

$$\lambda(S,t) = b(t) + c\,\sigma^2(S,t) = b(t) + c\,a^2(t)S^{2\beta},\ b(t) \geq 0,\ c > 0.$$

- Motivation for this class of $\lambda(S,t)$? Theoretical consistency?

- Peter Carr and V.L., "A Jump-to-Default Extended CEV Model: An Application of Bessel Processes," *Finance and Stochastics*, 10 (3), 303-330.

Solutions Under Carr and Wu

- Affine SV model with default and stochastic rates (extenstion of Carr & Wu (2005) with stochastic rates):

$$dS_t = (r_t - q + \lambda_t)S_t dt + \sqrt{V_t} S_t dW_t^S,$$
$$dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r \sqrt{r_t}\, dW_t^r,$$
$$dV_t = \kappa_V(\theta_V - V_t)dt + \sigma_V \sqrt{V_t}\, dW_t^V,$$
$$dz_t = \kappa_z(\theta_z + \gamma V_t - z_t)dt + \sigma_z \sqrt{z_t}\, dW_t^z,$$
$$\lambda_t = z_t + \alpha V_t + \beta r_t,$$
$$dW_t^S dW_t^V = \rho_{SV} dt, \ \rho_{SV} < 0,$$

  other correlations equal to zero.

- The model is affine and analytically tractable for European-style securities, incl. defaultable bonds and stock options, up to Fourier inversion.

Big Picture Questions and Extensions

1. Theoretical work is innovative. Solutions are neat.
2. Theoretical justification for intensities.
3. General properties of risk-neutral densities of equity returns when fitted to options.
4. Empirical work is needed to assess different models. What differentiates different models?
5. Credit risk model comparisons.